Closed-Form Caplet Pricing in the Black-Karasinski Short Rate Model

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Abstract

We present and analyse a beta blend model, a displaced diffusion-type model with volatility represented effectively as a combination of normal and lognormal components. The model contains the celebrated Hull-White ($\beta = 1$) and Black-Karasinski ($\beta = 0$) models as limiting cases. We conduct an asymptotic analysis in the limit of small volatility, utilising the perturbation expansion techniques proposed by Turfus and Schubert (2016) and Turfus (2016). We propose that the asymptotic formulae developed should be usable in practice for all values of $\beta \in [0, 1]$.

We calibrate the model so as to give consistent representation of the probability distribution of T-maturity conditional zero coupon bond prices at all future observation times up to maturity T. We then use the calibrated model to derive a second-order accurate representation of caplet prices, assuming a known, mean-reverting volatility term structure. Alternatively we can use this formula as the basis for calibrating the volatility term structure to known (ATM) caplet prices. The formula derived for $\beta < 1$ is seen to be closely related to the well-known Hull and White (1990) exact closed-form solution, but with a small (second order) convexity adjustment. The caplet formula derived here for the model of Black and Karasinski (1991) is believed to be presented here for the first time.

Key Words: caplet; Black-Karasinski; short rate model; closed form solution; perturbation method; asymptotic analysis.

1 Introduction

A short rate model with mean-reverting interest rates and lognormal volatitility was first proposed by Black and Karasinski (1991) twenty-five years ago. This model has obvious attractions, as noted by Brigo and Mercurio (2006):

> Since the market formulas for caps and swaptions are based on the assumption of lognormal rates, it seemed reasonable to choose the same distribution for the instantaneous short-rate process. Moreover, the rather good fitting quality of the model to market data, and especially to the swaption volatility surface, has made the model quite popular among practitioners and financial engineers.

However the model is known to have a significant drawback in that it is not analytically tractable. As Brigo and Mercurio (2006) continue:

> This renders the model calibration to market data more burdensome than in the Hull and White (1990) Gaussian model, since no analytical formulas for bonds are available. Indeed, when using a tree to price an option on a zero-coupon bond, one has to construct the tree until the bond maturity, which may actually be much longer than that of the option.

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For this reason, the affine models of Hull and White (1990) and Cox, Ingersoll and Ross (1985) have tended to be the most popular short rate models, certainly in terms of their dominating the published literature and probably also in terms of usage by the greatest number of practitioners.

Twenty-five years on, we seek to remedy this situation in the present paper. Following the approach of Turfus and Schubert (2016) and Turfus (2016), we use perturbation methods to find closed form solutions for conditional bond and caplet prices which, although approximate, should be of sufficient accuracy to be practically usable. The results are related to those obtained by Kim and Kunitomo (1999) for European equity option prices under stochastic rates. We embed our derivation in a general "beta blend" model framework which includes the Hull-White and Black-Karasinski models as special cases. We start off by deriving formulae for conditional bond prices, providing in the process analytic formulae for calibration of the model(s) without resort to numerical computation. These formulae are accurate under the assumption of small volatility for the interest rate process, with errors of third order for the bond price and of fourth order for the calibration. We go on to derive formulae for caplets and floorlets, with errors which we suggest are fourth order.

2 Modelling Assumptions

We consider the process r_t representing the interest short rate to be driven by a beta blend model, defined as follows. Rather than using r_t directly, we shall find it convenient to work with a reduced variable $\hat{x}_t$ satisfying the following canonical Ornstein-Uhlenbeck process:

$$d\hat{x}_t = -\alpha\hat{x}_t dt + \sigma_r(t)\, dW_t \tag{1}$$

where α is a positive constant,[1] $\sigma_r(t)$ is a bounded positive L^2 function and dW_t is a Brownian motion under the money market numéraire which we shall work with throughout. We shall suppose $t \in [t_0, T_m]$ for some finite T_m. Eq. (1) is well-known to have a unique strong solution subject to the stated assumptions given by

$$\hat{x}_t = \hat{x}_{t_0} e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-s)}\sigma_r(s)\, dW_s.$$

Without loss of generality we suppose that $\hat{x}_{t_0} = 0$.

The variable $\hat{x}_t$ is taken to be related to r_t by

$$(1-\beta)\, r_t + \beta\overline{r}(t) = (\overline{r}(t) + (1-\beta)r^*(t))\mathcal{E}\left(\frac{(1-\beta)\hat{x}_t}{|\overline{r}(t)|^\beta}\right) \tag{2}$$

with $\beta \in [0,1)$ assumed, $\overline{r}(t) > 0$ the instantaneous forward rate, determined with reference to market data and a suitable interpolation methodology[2] and $\mathcal{E}(.)$ a Doléans-Dade exponential defined by

$$\mathcal{E}(X_t) := \exp\left(X_t - \tfrac{1}{2}[X]_t\right) \tag{3}$$

with $[X]_t$ the quadratic variation of X_t, given for the case of $\hat{x}_t$ by $[\hat{x}]_t = I_r(t_0, t)$, where

$$I_r(t,v) := \int_t^v e^{-2\alpha(v-s)}\sigma_r^2(s)\, ds. \tag{4}$$

[1] It is a straightforward matter to loosen this assumption and specify instead that $\alpha(t)$ be a bounded positive L^1 function, whereupon the analysis below goes through effectively replacing $\alpha(v-u)$ throughout by $\int_u^v \alpha(s)ds$.

[2] We observe that, for $\beta \in (0,1)$, r_t is not defined at times t for which $\overline{r}(t) = 0$ if we do not constrain $\overline{r}(t)$ to be > 0. This problem can be mitigated by replacing $|\overline{r}(t)|^\beta$ on the denominator in Eq. (2) with $(|\overline{r}(t)| + (1-\beta)\delta)^\beta$ for some small smoothing parameter δ. This will ensure a positive denominator (even for negative rates) as well as maintaining Hull-White and Black-Karasinski as the limiting case models when $\beta \to 1$ and $\beta \to 0$ respectively. The analysis below goes through unaffected making this substitution throughout, effectively allowing the restriction that $\overline{r}(t) > 0$ to be lifted subject to this adjustment being made.

As can be seen, the beta blend model for intermediate values of β provides an interpolation between the well-known Hull and White (1990) normal model ($\beta \to 1$) and the Black and Karasinski (1991) lognormal model ($\beta = 0$). Indeed we shall find asymptotic solutions to the Hull-White model are recovered as the limit as $\beta \to 1^-$ of solutions based on Eq. (2). The function $r^*(t)$ is determined by calibration. We note however that, taking the expectation of Eq. (2) under the money market numéraire, it must satisfy

$$E[r_t] = \overline{r}(t) + r^*(t),$$

so is clearly constrained to tend to zero in the zero volatility limit. The formal no-arbitrage constraint which determines the functional form of $r^*(t)$ is as follows

$$E\left[e^{-\int_{t_0}^{t} r_s\,ds}\right] = e^{-\int_{t_0}^{t} \overline{r}(s)\,ds} \tag{5}$$

under the martingale measure (money market numéraire) for $t_0 < t \le T_m$, where T_m is the longest maturity date for which the model is calibrated. We proceed by writing the (stochastic) price at time t of a unit cash flow (zero coupon bond) at time T as $f_t^T = \hat{f}_T(\hat{x}_t, t)$. We then look to price these cash flows at time t_0 and so to determine the conditions necessary to satisfy Eq. (5), noting that $f_{t_0}^T = e^{-\int_{t_0}^{T} \overline{r}(s)\,ds}$.

For future notational convenience, we introduce at this stage the deterministic discount factors

$$D(t_1, t_2) = e^{-\int_{t_1}^{t_2} \overline{r}(s)\,ds}, \tag{6}$$

in terms of which we can also write $f_{t_0}^T = D(t_0, T)$.

3 Calibration of Model to Zero Coupon Bonds

We deduce by standard means that the T-maturity zero coupon bond price $\hat{f}_T(\hat{x}, t)$ will be governed under the money market numéraire by the following backward diffusion equation:

$$\frac{\partial \hat{f}_T}{\partial t} - \alpha\hat{x}\frac{\partial \hat{f}_T}{\partial \hat{x}} + \tfrac{1}{2}\sigma_r^2(t)\frac{\partial^2 \hat{f}_T}{\partial \hat{x}^2} = \frac{1}{1-\beta}\left((\overline{r}(t) + (1-\beta)r^*(t))\mathcal{E}\left(\frac{(1-\beta)\hat{x}_t}{|\overline{r}(t)|^\beta}\right) - \beta\overline{r}(t)\right)\hat{f}_T, \tag{7}$$

the r.h.s. being simply $\hat{f}_T$ multiplied by our assumed functional representation of r_t.

In the absence of exact closed form solutions to Eq. (7), we seek an approximate solution under a "weak volatility" assumption. To this end we rescale both $\hat{x}_t$ and $\sigma_x(t)$ by an asymptotic parameter ϵ defined by

$$\epsilon^2 := \frac{1}{\alpha(T_m - t_0)}\int_{t_0}^{T_m} \frac{\sigma_r^2(t)}{|\overline{r}(t)|^{2\beta}}dt$$

which we take to be small. Here we take T_m to be the longest maturity trade for which we wish our model to be calibrated. Thus we define new scaled variables x_t, $\sigma_x(t)$ and $I_x(t)$ by

$$\begin{aligned} x_t &:= \epsilon^{-1}\hat{x}_t, \\ \sigma_x(t) &:= \epsilon^{-1}\sigma_r(t), \\ I_x(t,v) &:= \epsilon^{-2}I_r(t,v), \end{aligned}$$

all taken to be $O(1)$ as $\epsilon \to 0$. We further define a new functional form for f_t^T in terms of the new co-ordinate x_t:

$$f_t^T = f_T(x_t, t).$$

Performing the rescaling we obtain

$$\frac{\partial f_T}{\partial t} - \alpha x\frac{\partial f_T}{\partial x} + \tfrac{1}{2}\sigma_x^2(t)\frac{\partial^2 f_T}{\partial x^2} = \frac{1}{(1-\beta)}\left((\overline{r}(t) + (1-\beta)r^*(t))\mathcal{E}\left(\frac{\epsilon(1-\beta)x_t}{|\overline{r}(t)|^\beta}\right) - \beta\overline{r}(t)\right)f_T, \tag{8}$$

in place of the original Eq. (7). On this basis we propose an asymptotic expansion

$$r^*(t) = \epsilon r_1^*(t) + \epsilon^2 r_2^*(t) + \epsilon^3 r_3^*(t) + O(\epsilon^4), \tag{9}$$

with the $r_i^*(.)$ to be determined. We can then rewrite Eq. (8) as

$$\mathcal{L}[f_T(x,t)] = h(x,t) f_T(x,t) \tag{10}$$

where $\mathcal{L}[.]$ is a standard forced diffusion operator given by

$$\mathcal{L}[.] = \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} + \frac{1}{2}\sigma_r^2(t)\frac{\partial^2}{\partial x^2} - \overline{r}(t), \tag{11}$$

and the (asymptotically small) forcing function is given by

$$\begin{aligned} h(x,t) &:= \frac{1}{1-\beta}\left((\overline{r}(t) + (1-\beta) r^*(t))\mathcal{E}\left(\frac{\epsilon(1-\beta)x_t}{|\overline{r}(t)|^\beta}\right) - \overline{r}(t)\right) \\ &= \epsilon h_1(x,t) + \epsilon^2 h_2(x,t) + \epsilon^3 h_3(x,t) + O(\epsilon^4). \end{aligned} \tag{12}$$

By expanding the exponential in Eq. (12) as a power series about $\epsilon = 0$ and gathering together like powers of ϵ, we conclude

$$h_1(x,t) = \frac{\overline{r}(t)}{|\overline{r}(t)|^\beta} x + r_1^*(t), \tag{13}$$

$$h_2(x,t) = \tfrac{1}{2}(1-\beta)\frac{\overline{r}(t)}{|\overline{r}(t)|^{2\beta}}(x^2 - I_x(t_0,t)) + (1-\beta)\frac{r_1^*(t)}{|\overline{r}(t)|^\beta} x + r_2^*(t), \tag{14}$$

$$h_3(x,t) = \tfrac{1}{6}(1-\beta)^2\frac{\overline{r}(t)}{|\overline{r}(t)|^{3\beta}} x^3 + \tfrac{1}{2}(1-\beta)^2\frac{r_1^*(t)}{|\overline{r}(t)|^{2\beta}}(x^2 - I_x(t_0,t)) + (1-\beta)\frac{r_2^*(t)}{|\overline{r}(t)|^\beta} x + r_3^*(t), \tag{15}$$

The final condition that must be satisfied is

$$f_T(x_T, T) = 1. \tag{16}$$

Writing the zero coupon bond price as a perturbation expansion

$$f_T(x,t) = f_0(t,T) + \epsilon f_1(x,t,T) + \epsilon^2 f_2(x,t,T) + O(\epsilon^3), \tag{17}$$

and substituting into Eq. (10) we are able to obtain an approximate solution by standard means.

Theorem 3.1 *The zero coupon bond price satisfying Eq. (8) subject to the final condition Eq. (16) is given in asymptotic form by Eq. (17) with*

$$\begin{aligned} f_0(t,T) &= D(t,T) \\ f_1(x,t,T) &= -D(t,T) x B_1^*(t,T) \\ f_2(x,t,T) &= D(t,T)\left((x^2 - I_x(t_0,t))\left(\tfrac{1}{2}B_1^*(t,T)^2 - B_2^*(t,T)\right) - I_x^{(1)}(t) B_1^*(t,T)\right) \end{aligned}$$

where

$$B_j^*(t_1,t_2) := \frac{(1-\beta)^{j-1}}{j!}\int_{t_1}^{t_2} e^{-j\alpha(u-t_1)}\frac{\overline{r}(u)}{|\overline{r}(u)|^{j\beta}} du, \tag{18}$$

$$I_x^{(1)}(t) := \int_{t_0}^{t} e^{-\alpha(t-u)} I_x(t_0,u)\frac{\overline{r}(u)}{|\overline{r}(u)|^\beta} du. \tag{19}$$

Proof. For the proof of Theorem 3.1, see Appendix A. □

Proposition 3.1 *The zero coupon bond price presented in Theorem 3.1 can conveniently be re-expressed to the same level of approximation as*[3]

$$f_T(x_t,t) = f_T^*(x_t,t) - \epsilon^2 D(t,T)\left(x_t^2 - I_x(t_0,t)\right) B_2^*(t,T) + O(\epsilon^3) \tag{20}$$

where

$$f_T^*(x_t,t) := D(t,T)\mathcal{E}\left(-(\epsilon x_t + \epsilon^2 I_x^{(1)}(t))B_1^*(t,T)\right) \tag{21}$$

The formulation in Eqs. (20)-(21) is motivated by the fact that it expresses the solution as a perturbation about a lognormal distribution, thus facilitating the use of standard Black-type formulae in the analysis, as will be seen in the following section. It furthermore emphasises the connection that in the Hull-White $\beta \to 1$ limit, wherein $f_T^*(x_t,t)$ is seen to be the exact solution under the money market numeraire.

Corollary 3.1 *Satisfaction of the no-arbitrage condition Eq. (5) for r_t requires that in Eq. (9) we choose $r_1^*(t) \equiv r_3^*(t) \equiv 0$ and*

$$r_2^*(t) = \frac{\overline{r}(t)}{|\overline{r}(t)|^\beta} I_x^{(1)}(t).$$

Further, we infer from Eqs. (13)-(15) that

$$\begin{aligned}
h_1(x,t) &= \frac{\overline{r}(t)}{|\overline{r}(t)|^\beta}x, \\
h_2(x,t) &= \tfrac{1}{2}(1-\beta)\frac{\overline{r}(t)}{|\overline{r}(t)|^{2\beta}}(x^2 - I_x(t_0,t)) + r_2^*(t), \\
h_3(x,t) &= \tfrac{1}{6}(1-\beta)^2\frac{\overline{r}(t)}{|\overline{r}(t)|^{3\beta}}x^3 + (1-\beta)r_2^*(t)\frac{\overline{r}(t)}{|\overline{r}(t)|^\beta}x.
\end{aligned}$$

Proof. These results arise naturally as a by-product of the proof of Theorem 3.1. □

This completes the calibration of our model to $O(\epsilon^3)$ accuracy, with errors of $O(\epsilon^4)$. It is a straightforward, albeit lengthy, matter on this basis to extend the expansion in Eq. (17) to include $f_3(x,t)$. However the above results are adequate for the present purpose of calculating caplet prices to 2nd/3rd order accuracy.

4 Caplet Pricing

We next consider the calibration of the volatility term structure $\sigma_r(t)$ of our interest rate model to the caplet market. We consider only one caplet per maturity peg, it not being possible to calibrate the beta blend model to a volatility skew or smile without assuming a local (or stochastic) volatility, which task is beyond the scope of the present calculation. To carry out the calibration we seek closed form solutions for caplet prices consistent with the asymptotic approximation scheme proposed above.[4]

[3] We note here and below that, in specifying the stochastic process x_t as an argument rather than the variable x, we are defining the relevant function on a random curve in (x,t) space rather than in the space itself. This is mainly for notational convenience. If wished, the Doléans-Dade exponentials which appear can be expanded by writing the required quadratic variations out in full, so recovering a form of solution in which x_t is interchangeable with x.

[4] The alternative without such closed form solutions would typically be to roll out a new Monte Carlo simulation of Eq. (1) for each maturity peg and employ an iterative scheme to adjust the volatility level until the required caplet price is reproduced. This approach faces the further difficulty of the need to compute for each simulated scenario the numéraire $e^{\int_{t_0}^{T-\tau} r_s ds}$ required to discount the simulated payoff. Alternatively, a PDE approach could be employed, avoiding the need for the discount factor calculation. However, given that a common regulatory requirement these days is for banks regularly to reprice their portfolios under multiple market scenarios, the high cost of such a calibration strategy for interest rate volatility would have to be seen as a significant deterrent to adoption of beta blend models. Indeed, it would appear that, among short rate models, only affine models (Hull-White and Cox-Ingersoll-Ross) which do have (exact) closed form caplet formulae are widely used.

4.1 Payoff Modelling

Consider a caplet which pays the positive difference between tenor-τ Libor and a strike K on a unit notional, based on a payment period $[T-\tau, T]$, which rate we shall denote $L(\tau, T)$. For simplicity we assume no spread between forward Libor rates and the equivalent risk-free rates inferred from Eq. (2) above. However, it is not difficult to introduce an assumed deterministic spread by adjusting the value of the strike accordingly in the formulae derived below. We denote the (stochastic) value at time t of the caplet by $C_{T,K}(x,t)$. We can express the payoff at time T by

$$\text{Payoff}_T := \max\{(L(\tau,T) - K)\delta(T-\tau,T), 0\}$$

where $\delta(t_1, t_2)$ is the day count faction calculated according to the relevant convention (usually actual/360 or actual/365). We note in particular that, under our assumptions, the realised Libor rate is related to the stochastic zero coupon bond price $f_T(x,t)$ calculated above by

$$1 + L(\tau,T)\delta(T-\tau,T) = \frac{1}{f_T(x, T-\tau)} \tag{22}$$

whence

$$\text{Payoff}_T(x) = \max\left\{f_T(x, T-\tau)^{-1} - (1 + K\delta(T-\tau,T)), 0\right\}$$

If we consider an equivalent payoff payment made at time $T-\tau$, this must be discounted by precisely the T-maturity zero coupon bond price observed at time $T-\tau$, whence we can write

$$\text{Payoff}_{T-\tau}(x) = \kappa^{-1}\max\{\kappa - f_T(x, T-\tau), 0\} \tag{23}$$

where

$$\kappa := \frac{1}{1 + K\delta(T-\tau,T))}. \tag{24}$$

In other words we should consider a put option on the bond price. See Brigo and Mercurio (2006) for more details of the application of this approach to affine short rate models.

Writing the price of this option as $C_{T,K}(x,t)$, we see this will satisfy

$$\mathcal{L}[C_{T,K}(x,t)] = h(x,t)C_{T,K}(x,t), \tag{25}$$

with $\mathcal{L}[.]$ and $h(x,t)$ given by Eqs. (11) and (12) above. The final condition satisfied will be

$$C_{T,K}(x, T-\tau) = \text{Payoff}_{T-\tau}(x). \tag{26}$$

We note in passing that, if we remove the max condition from Eq. (23) and use instead

$$\text{Payoff}_T := (L(\tau,T) - K)\delta(T-\tau,T),$$

we obtain the price of a short position in a Libor forward rate agreement, which we denote $F_{T,K}(x,t)$. Standard no-arbitrage arguments show this to be given by

$$F_{T,K}(x,t) = f_{T-\tau}(x,t) - \kappa^{-1} f_T(x,t) \tag{27}$$

which can be evaluated asymptotically making use of Eq. (20). Also, setting $\kappa = 1$ in the above yields the t-value of the forward Libor contract.

4.2 Perturbation Analysis

To solve for $C_{T,K}(0, t_0)$, we pose formally:[5]

$$C_{T,K}(x,t) = C_0(x,t) + \epsilon C_1(x,t) + \epsilon^2 C_2(x,t) + O(\epsilon^3). \tag{28}$$

Although we choose not to make it explicit in our notation, each of the $C_i(x,t)$ is expected to have a weak dependence on ϵ but to be bounded independently of ϵ as $\epsilon \to 0$. Substituting the above expansion into Eq. (25) and proceeding as previously we obtain:

[5]For notational convenience, we drop the explicit dependence of the $C_i(.)$ on K and T, treating them as fixed parameters.

Theorem 4.1 *The price of a caplet fixing at time $T-\tau$ for payment of* Payoff_T *at time T is given in original unscaled notation by*

$$C_{T,K}(0,t_0) = D(t_0,T-\tau)N(-d_2) - \kappa^{-1}D(t_0,T)\left(N(-d_1) + I_r(t_0,T-\tau)\,B_2^*(T-\tau,T)d_1N'(-d_1)\right) + O(\epsilon^3) \tag{29}$$

where $N(.)$ is a unit normal cumulative distribution function and we have defined

$$d_1 := \frac{\ln\left(\kappa^{-1}D(T-\tau,T)\right) + \frac{1}{2}B_1^*(T-\tau,T)^2 I_r(t_0,T-\tau)}{B_1^*(T-\tau,T)\sqrt{I_r(t_0,T-\tau)}}, \tag{30}$$

$$d_2 := d_1 - B_1^*(T-\tau,T)\sqrt{I_r(t_0,T-\tau)}. \tag{31}$$

Proof. For the proof of Theorem 4.1, see Appendix B. □

As can be seen, Eq. (29) takes the form of the standard Black formula with an asymptotically small adjustment, effectively at $O(\epsilon^2)$. Notably, there is no explicit dependence on β, this parameter influencing the form of the result only through its impact on the $B_j^*(.)$.

We observe that, although we formally excluded the case $\beta = 1$ from consideration as it is a singular limit, the formula none the less remains coherent in that limit and indeed the well-known Hull-White limit formula is recovered to the level of approximation considered (effectively by setting $B_2^*(.)$ to zero).

As a further note, although it might be argued for the lognormal case ($\beta = 0$) that the local volatility (more specifically $I_r(t_0,T-\tau)$) is unlikely to be particularly small in practice, we suggest that this is not as much of an issue as might appear, since everywhere the "small" factor $I_r(t_0,t)$ appears in formulae, it is in the context of a bilinear functional of the short rate (itself a small parameter), effectively scaled in such circumstances by τ, through $B_j^*(T-\tau,T)$ coefficients.[6] We suggest that, even when $I_r(t_0,t)$ is not particularly small, its impact once scaled down in this way can in most practical circumstances be expected to be small enough for our approximation to be usable. Although we have not formally proved it, symmetry considerations suggest that $C_3(x,t) = O(x)$ as $x \to 0$, whence the next correction to Eq. (29) will be at $O(\epsilon^4)$, rather than $O(\epsilon^3)$. Further work is under way to explore the accuracy of the above-proposed second order solution.

Finally, we can also state:

Corollary 4.1 *The corresponding floorlet price is*

$$F_{T,K}(0,t_0) = \kappa^{-1}D(t_0,T)\left(N(d_1) - I_r(t_0,T-\tau)\,B_2^*(T-\tau,T)d_1N'(d_1)\right) - D(t_0,T-\tau)N(d_2) + O(\epsilon^3). \tag{32}$$

Proof. This follows immediately from put-call parity. □

A Proof of Theorem 3.1

Substituting from Eq. (17) into Eq. (8) and equating terms we find at zeroth order that $\mathcal{L}[f_0(t)] = 0$ with the trivial solution that

$$f_0(t,T) = D(t,T)$$

(see Eq. (6) above), which clearly satisfies the required final condition. At first order we have

$$\mathcal{L}[f_1(x,t,T)] = h_1(x,t)f_0(t,T) \tag{33}$$

The final condition Eq. (16) is satisfied if $f_1(x_T,T,T) = 0$. The no-arbitrage condition Eq. (5) introduces the further constraint that $f_1(0,t_0,T) = 0$.

[6] In other words, it is the smallness of the volatility of r_t, not of x_t which is required to be small.

To proceed we observe that Eq. (11) has a Green's function solution given by

$$G(x,t;\xi,v) = \frac{D(t,v)H(v-t)}{\sqrt{I_x(t,v)}} N'\left(\frac{xe^{-\alpha(v-t)} - \xi}{\sqrt{I_x(t,v)}}\right), \tag{34}$$

where $H(.)$ is the Heaviside step function and $N(x)$ is a unit normal cumulative distribution function. We deduce by standard means that:

$$f_1(x,t,T) = -D(t,T)\left(xB_1^*(t,T) + \int_t^T r_1^*(v)\,dv\right).$$

with $B_1^*(t,T)$ given by Eq. (18). Clearly satisfaction of the no-arbitrage condition $f_1(0,t_0,T) = 0$ requires that we take $r_1^*(.) \equiv 0$, whence we conclude

$$f_1(x,t,T) = -D(t,T)xB_1^*(t,T), \tag{35}$$

which clearly also satisfies the required final condition.

Proceeding in a similar vein, we find at second order

$$\mathcal{L}[f_2(x,t,T)] = h_1(x,t)f_1(x,t,T) + h_2(x,t)f_0(t,T) \tag{36}$$

with solution

$$\begin{aligned} f_2(x,t,T) &= -\int_t^T \int_{-\infty}^{\infty} G(x,t;\xi,v)(h_1(\xi,v)f_1(\xi,v,T) + h_2(\xi,v)f_0(v,T))\,d\xi\,dv \\ &= D(t,T)\int_t^T \left[\left(x^2e^{-2\alpha(v-t)} + I_x(t,v)\right)\frac{\overline{r}(v)}{|\overline{r}(v)|^\beta}B_1^*(v,T) - \tfrac{1}{2}(1-\beta)\frac{\overline{r}(v)}{|\overline{r}(v)|^{2\beta}}x^2e^{-2\alpha(v-t)} - r_2^*(v)\right]dv, \end{aligned}$$

which clearly satisfies the required final condition that $f_2(x_T,T,T) = 0$.

At $t = t_0$ we have

$$f_2(0,t_0,T) = D(t_0,T)\int_{t_0}^T \left(\frac{\overline{r}(v)}{|\overline{r}(v)|^\beta}I_x^{(1)}(v) - r_2^*(v)\right)dv \tag{37}$$

where $I_x^{(1)}(.)$ is given by Eq. (19)and we have used Eq. (18) and inverted the order of integration between u and v. Satisfaction of the no-arbitrage condition $f_2(0,t_0,T) = 0$ is then achieved by choosing[7]

$$r_2^*(t) = \frac{\overline{r}(t)}{|\overline{r}(t)|^\beta}I_x^{(1)}(t). \tag{38}$$

Clearly for a given representation of the term structure $\overline{r}(t)$ of interest rates and $\sigma_r(t)$ of interest rate volatility, Eq. (38) allows the model to be calibrated accurate to $O(\epsilon^2)$. The observation that successive terms in the expansions for $r^*(.)$ and $h(.)$ are by construction alternately odd then even functions of x allows us further to infer that $r_3^*(.) \equiv 0$. The form of the $h_i(.)$ are then readily obtainable from Eqs. (13)-(15). Our expansion will therefore in practice give rise to calibration errors only at $O(\epsilon^4)$. We further deduce[8]

$$f_2(x,t,T) = D(t,T)\left(\left(x^2 - I_x(t_0,t)\right)\left(\tfrac{1}{2}B_1^*(t,T)^2 - B_2^*(t,T)\right) - I_x^{(1)}(t)B_1^*(t,T)\right). \tag{39}$$

This completes the proof of Theorem 3.1. □

[7] We note that, following the reversal of the order of integration, the parameter T does not appear in the expression inferred for $r_2^*(t)$, as would be expected since it relates to the model, not to the particulars of the cash flow being priced.

[8] We have here used Eq. (18) with $j = 1$ and the identity that, by symmetry, $\int_t^T f(v)\int_t^v f(u)\,du\,dv = \frac{1}{2}\left(\int_t^T f(u)\,du\right)^2$.

B Proof of Theorem 4.1

Substituting our expansion Eq. (28) into Eq. (25), at leading order we have

$$\mathcal{L}[C_0(x,t)] = 0. \tag{40}$$

Since the functional form of the underlying in the payoff Eq. (23) is provided only as a perturbation expansion, the final condition Eq. (26) cannot be applied directly. Writing for notational convenience

$$\begin{aligned} P_0(\xi) &= \kappa^{-1}(\kappa - f_T^*(\xi, T-\tau)), \\ P_2(\xi) &= \kappa^{-1} D(T-\tau, T)\left(\xi^2 - I_x(t_0, T-\tau)\right) B_2^*(T-\tau, T), \end{aligned}$$

with $B_j^*(.)$ and $I_x^{(1)}(.)$ as defined in Eqs. (18) and (19) respectively, the payoff can be expressed as

$$\text{Payoff}_{T-\tau}(x) \sim P_0(x) + \epsilon^2 P_2(x))H(P_0(x) + \epsilon^2 P_2(x))$$

We rewrite this as

$$\begin{aligned} \text{Payoff}_{T-\tau}(x) \sim\ & P_0(x)H(P_0(x)) \\ & + \epsilon^2 P_2(x)H(P_0(x)) + P_0(x)(H(P_0(x) + \epsilon^2 P_2(x)) - H(P_0(x))) \\ & + \epsilon^2 P_2(x)(H(P_0(x) + \epsilon^2 P_2(x) - P_0(x)) \end{aligned}$$

where we expect the first term on the r.h.s. to drive contributions at zeroth order and the terms in the second line contributions at $O(\epsilon^2)$. The last line, yielding expected contributions at $O(\epsilon^4)$, we can neglect in the context of the present second order analysis. This leads us to pose the following final condition for the leading order solution:

$$C_0(x, T-\tau) = P_0(x)H(P_0(x)).$$

For notational convenience we write the root of $P_0(x)$ as

$$x^* = \frac{\ln\left(D(T-\tau,T)\kappa^{-1}\right)}{\epsilon B_1^*(T-\tau,T)} - \tfrac{1}{2}\epsilon B_1^*(T-\tau,T) I_x(t_0, T-\tau) - \epsilon I_x^{(1)}(T-\tau).$$

Using the Green's function Eq. (34), we deduce that

$$\begin{aligned} C_0(x,t) &= \frac{D(t,T-\tau)}{\sqrt{I_x(t,T-\tau)}} \int_{x^*}^{\infty} P_0(\xi) N'\left(\frac{\xi - x e^{-\alpha(T-\tau-t)}}{\sqrt{I_x(t,T-\tau)}}\right) d\xi \\ &= D(t,T-\tau)\left(N(-d_2^*(x,t)) - \kappa^{-1} F_T^*(x,t,T-\tau) N(-d_1^*(x,t))\right) \end{aligned} \tag{41}$$

with

$$F_T^*(x,t,u) := D(u,T)\exp\Big(-\epsilon x e^{-\alpha(u-t)} B_1^*(u,T) - \tfrac{1}{2}\epsilon^2 e^{-2\alpha(u-t)} B_1^*(u,T)^2 I_x(t_0,t) - \epsilon^2 B_1^*(u,T) I_x^{(1)}(u)\Big), \tag{42}$$

$$d_2^*(x,t) := \frac{\ln\left(\kappa^{-1} F_T^*(x,t,T-\tau)\right) - \tfrac{1}{2}\epsilon^2 B_1^*(T-\tau,T)^2 I_x(t,T-\tau)}{\epsilon B_1^*(T-\tau,T)\sqrt{I_x(t,T-\tau)}}, \tag{43}$$

$$d_1^*(x,t) := d_2^*(x,t) + \epsilon B_1^*(T-\tau,T)\sqrt{I_x(t,T-\tau)}. \tag{44}$$

At first order, we must then solve

$$\mathcal{L}[C_1(x,t)] = h_1(x,t) C_0(x,t), \tag{45}$$

with $h_1(x,t)$ as given in Corollary 3.1 and final condition $C_1(x, T-\tau) = 0$. Applying the same methodology as previously, we obtain[9]

$$
\begin{aligned}
C_1(x,t) &= -\int_t^{T-\tau} \frac{D(t,v)}{\sqrt{I_x(t,v)}} \frac{\overline{r}(v)}{|\overline{r}(v)|^\beta} \int_{-\infty}^{\infty} \xi\, C_0(\xi, v) N'\left(\frac{\xi - xe^{-\alpha(v-t)}}{\sqrt{I_x(t,v)}}\right) d\xi\, dv \\
&= -D(t, T-\tau) \int_t^{T-\tau} \frac{\overline{r}(v)}{|\overline{r}(v)|^\beta} \left(xe^{-\alpha(v-t)} - \frac{I_x(t,v)}{e^{-\alpha(v-t)}} \frac{\partial}{\partial x}\right) \\
&\qquad \Big(N(-d_2^*(xe^{-\alpha(v-t)}, t)) - \kappa^{-1} F_T^*(x, t, T-\tau) N(-d_1^*(xe^{-\alpha(v-t)}, t))\Big)\, dv. \qquad (46)
\end{aligned}
$$

Setting $t = t_0$ and noting that

$$
F_T^*(0, t_0, T-\tau) = D(T-\tau, T) \exp(-\epsilon^2 B_1^*(T-\tau, T) I_x^{(1)}(T-\tau))
$$

we obtain

$$
C_1(0, t_0) = -\epsilon D(t_0, T) \kappa^{-1} B_1^*(T-\tau, T) I_x^{(1)}(T-\tau) N(-d_1^*(0, t_0)) + O(\epsilon^3). \qquad (47)
$$

It is convenient to make use at this point of the fact that $d_i^*(0, t_0) = d_i + O(\epsilon^2)$ to carry out an expansion of Eq. (41) and combine the zeroth and first order contributions to the caplet price as follows:

$$
C_0(0, t_0) + \epsilon C_1(0, t_0) = D(t_0, T-\tau) N(-d_2) - \kappa^{-1} D(t_0, T) N(-d_1) + O(\epsilon^4). \qquad (48)
$$

At second order, we then must satisfy

$$
\mathcal{L}[C_2(x,t)] = h_2(x,t) C_0(x,t) + h_1(x,t) C_1(x,t) \qquad (49)
$$

We note here that, for purposes of computing $C_2(x,t)$, the impact of contributions to $C_0(x,t)$ and $C_1(x,t)$ at $O(\epsilon)$ and higher in Eq. 49 can be ignored, whence we propose the use of the following leading order estimates:

$$
\begin{aligned}
\tilde{C}_0(t) &= D(t, T-\tau) \left(N(-d_2) - \kappa^{-1} D(T-\tau, T) N(-d_1)\right), \\
\tilde{C}_1(x,t) &= -x\, \tilde{C}_0(t) \int_t^{T-\tau} \frac{\overline{r}(v)}{|\overline{r}(v)|^\beta} e^{-\alpha(v-t)} dv.
\end{aligned}
$$

On this basis, we seek to solve instead for an asymptotically equivalent representation $\tilde{C}_2(x,t) = C_2(x,t) + O(\epsilon)$ satisfying

$$
\mathcal{L}[\tilde{C}_2(x,t)] = h_2(x,t) \tilde{C}_0(t) + h_1(x,t) \tilde{C}_1(x,t) \qquad (50)
$$

In specifying the final condition applicable at second order, let us write the root of $P_0(x) + \epsilon^2 P_2(x)$ as $x^* + \epsilon \Delta x_1^*$, with $\Delta x_1^* = O(1)$. On this basis, we can make the following representation:

$$
H\left(P_0(x) + \epsilon^2 P_2(x)\right) \equiv H(x - x^* - \epsilon \Delta x_1^*),
$$

in terms of which the final condition for Eq. (50) can be written

$$
\tilde{C}_2(x, T-\tau) = P_2(x) H(x - x^*) + \epsilon^{-2} P_0(x) (H(x - x^* - \epsilon \Delta x_1^*) - H(x - x^*))
$$

We look for a solution in the form

$$
\tilde{C}_2(x,t) = C_2^{(0)}(x,t) + C_2^{(1)}(x,t)
$$

[9] In obtaining this result it is helpful to note that

$$
\xi\, N'\left(\frac{\xi - z}{\sigma}\right) = \left(z + \sigma^2 \frac{\partial}{\partial z}\right) N'\left(\frac{\xi - z}{\sigma}\right).
$$

where $C_2^{(0)}(x,t)$ satisfies the homogeneous Eq. (40), with the specified *nonhomogeneous* final condition, while $C_2^{(1)}(x,t)$ satisfies Eq. (50) but with a homogeneous final condition. The solution for $C_2^{(0)}(x,t)$ can be obtained as

$$\begin{aligned} C_2^{(0)}(x,t) &= \frac{D(t,T-\tau)}{\sqrt{I_x(t,T-\tau)}}\int_{x^*}^{\infty} P_2(\xi)N'\left(\frac{\xi - xe^{-\alpha(T-\tau-t)}}{\sqrt{I_x(t,T-\tau)}}\right)d\xi \\ &\quad + \epsilon^{-2}\frac{D(t,T-\tau)}{\sqrt{I_x(t,T-\tau)}}\int_{x^*+\epsilon\Delta x_1^*}^{x^*} P_0(\xi)N'\left(\frac{\xi - xe^{-\alpha(T-\tau-t)}}{\sqrt{I_x(t,T-\tau)}}\right)d\xi \\ &= \kappa^{-1}D(t,T)\Big[e^{-2\alpha(T-\tau-t)}\left(x^2 - I_x(t_0,t)\right)B_2^*(T-\tau,T)N(-d_2^*(x,t)) \\ &\quad + \Big(2xe^{-\alpha(T-\tau-t)}\sqrt{I_x(t,T-\tau)} - d_2^*(x,t)I_x(t,T-\tau)\Big)B_2^*(T-\tau,T)N'(-d_2^*(x,t))\Big] + O(\epsilon) \end{aligned} \tag{51}$$

where the second integral is found, on expanding in terms of the small parameter $\epsilon\Delta x_1^*$, to be $O(\epsilon^3)$, hence contributes only at $O(\epsilon)$. Setting $t = t_0$ we obtain

$$C_2^{(0)}(0,t_0) = -\kappa^{-1}D(t_0,T)I_x(t_0,T-\tau)B_2^*(T-\tau,T)d_2N'(-d_2) + O(\epsilon) \tag{52}$$

Finally we must solve Eq. (50) subject to a homogeneous final condition for $C_2^{(1)}(x,t)$. The calculation parallels that used to obtain Eq. (39) in the previous section. We obtain:

$$\begin{aligned} C_2^{(1)}(x,t) &= -\int_t^{T-\tau}\frac{D(t,v)}{\sqrt{I_x(t,v)}}\int_{-\infty}^{\infty}\Big(h_2(\xi,v)\tilde{C}_0(\xi,v) + h_1(\xi,v)\tilde{C}_1(\xi,v)\Big)N'\left(\frac{\xi - xe^{-\alpha(v-t)}}{\sqrt{I_x(t,v)}}\right)d\xi\,dv \\ &= -\tilde{C}_0(x,t)\Big(\left(x^2 - I_x(t_0,t)\right)\left(\tfrac{1}{2}B_1^*(t,T-\tau)^2 - B_2^*(t,T-\tau)\right) - I_x^{(1)}(t)B_1^*(t,T-\tau)\Big) + O(\epsilon). \end{aligned} \tag{53}$$

Clearly this vanishes at $t = t_0$, hence gives no contribution at second order to $C_{T,K}(0,t_0)$.

Combining the zeroth order contribution of Eq. (48) with the $O(\epsilon^2)$ contribution from Eq. (52) and reverting to our original unscaled notation, we deduce the result in Eq. (29). □

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